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Generalized Weighted Non-Symmetric Divergences and Their Bounds

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Abstract

We first develop these measures in this paper before studying the generalised weighted symmetric and non-symmetric divergences. Afterwards, using Csiszar's f-divergence finding as a springboard, examples of these divergence functions are shown while also taking into account their convexity and non-negativity. Then, we look at theorems that represent bounds on symmetric weighted divergence measures.

Keywords: "Relative weighted information of types: Weighted J-divergence, Weighted JS-divergence, Weighted AG-divergence, Csiszar's f-divergence."

1. Introduction

Let $$\Gamma_n = \begin{cases} P = (p_1, p_2, \ldots, p_n), p_i > 0, & \sum_{i=1}^{n} p_i = 1 \\ Q = (q_1, q_2, \ldots, q_n), q_i > 0, & \sum_{i=1}^{n} q_i = 1 \end{cases} \qquad n \geq 2$$

be the set of all complete finite 'discrete probability distributions. In light of Kullback Leibler's (1951) [19] relative information, weighted relative information is then considered as follows:

$$K(P; Q; W) = \sum_{i=1}^{n} w_i\, p_i \log\left(\frac{p_i}{q_i}\right), for\ all\ \ P, Q \in \Gamma_n \tag{1.1}$$

where $w_i > 0$, $\forall$ i = 1 to n the weighted distribution corresponding to probability distribution.

We observe that (1.1) is not symmetric in P & Q. Its symmetric version famous as "weighted J-divergence (Jeffreys [18]; Kullback and Leiber [19])" is given by

$$J((P; Q; W) = K((P; Q; W) + K(Q; P; W) = \sum_{i=1}^{n} w_i\, (p_i - q_i) \log\frac{p_i}{q_i} \tag{1.2}$$

For ease of use, we refer to the measurements K ($P; Q; W$) here as the adjoint of K ($Q; P; W$), and vice versa.

As an alternative, the measure J (Ᵽ; Ọ; W) may alternatively be expressed as follows:

$$J(Ᵽ; Ọ; W) = \mathcal{D}(Ᵽ; Ọ; W) + \mathcal{D}(Ọ; Ᵽ; W) \quad (1.3)$$

here

$$\mathcal{D}(Ᵽ; Ọ; W) = \sum_{i=1}^{n} w_i (p_i - q_i) \log\left(\frac{p_i + q_i}{2q_i}\right) \quad (1.4)$$

&

$$\mathcal{D}(Ọ; Ᵽ; W) = \sum_{i=1}^{n} w_i (q_i - p_i) \log\left(\frac{p_i + q_i}{2p_i}\right) \quad (1.5)$$

Take into account these two measures:

$$F(Ᵽ; Ọ; W) = K\left(Ᵽ; \frac{Ᵽ+Ọ}{2}; W\right) = \sum_{i=1}^{n} w_i\, p_i \log\left(\frac{2p_i}{p_i + q_i}\right) \quad (1.6)$$

&

$$G(Ᵽ; Ọ; W) = K\left(\frac{Ᵽ+Ọ}{2}; Ọ; W\right) = \sum_{i=1}^{n} w_i \left(\frac{p_i + q_i}{2}\right) \log\left(\frac{p_i + q_i}{2p_i}\right) \quad (1.7)$$

The measures (1.6) and (1.7)'s adjoint forms are provided by

$$F(Ọ; Ᵽ; W) = K\left(Ᵽ; \frac{Ᵽ+Ọ}{2}; W\right) = \sum_{i=1}^{n} w_i\, q_i \log\left(\frac{2q_i}{p_i + q_i}\right) \quad (1.8)$$

&

$$G(Ọ; Ᵽ; W) = K\left(\frac{Ᵽ+Ọ}{2}; Ọ; W\right) = \sum_{i=1}^{n} w_i \left(\frac{p_i + q_i}{2}\right) \log\left(\frac{p_i + q_i}{2q_i}\right) \quad (1.9)$$

respectively, and the symmetric forms are provided by

$$I(Ᵽ; Ọ; W) = \frac{1}{2}[F(Ᵽ; Ọ; W) + F(Ọ; Ᵽ; W)]$$
$$= \frac{1}{2}\left[\sum_{i=1}^{n} w_i\, p_i \log\left(\frac{2p_i}{p_i + q_i}\right) + \sum_{i=1}^{n} w_i\, q_i \log\left(\frac{2q_i}{p_i + q_i}\right)\right] \quad (1.10)$$

&

$$T(Ᵽ; Ọ; W) = \frac{1}{2}[G(Ᵽ; Ọ; W) + G(Ọ; Ᵽ; W)]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{n} w_i \left(\frac{þ_i + q_i}{2}\right) log\left(\frac{þ_i + q_i}{2\sqrt{þ_i q_i}}\right)\right] \quad (1.11)$$

respectively.

The relationships between the three measures J (Ᵽ; Ọ; W), I(Ᵽ; Ọ; W), and T(Ᵽ; Ọ; W) are as follows:

$$J(Ᵽ; Ọ; W) = 4[I(Ᵽ; Ọ; W) + T(Ᵽ; Ọ; W)] \quad (1.12)$$

Furthermore, the measures (1.4) can alternatively be expressed as

$$\mathcal{D}(Ᵽ; Ọ; W) = 2[F(Ọ; Ᵽ; W) + G(Ọ; Ᵽ; W)] \quad (1.13)$$

The weighted information radius (Sibson [27]), “weighted Jensen difference divergence measure”, or simply “weighted J.S. divergence” are popular names for the measure J(Ᵽ; Ọ; W) in the literature (Burbea and Rao [2, 3]). Without taking into account weighted distribution, Taneja [31] explored the measure (1.9) for the first time, and whenever weighted distribution is taken into account, we refer to it as “weighted arithmetic and geometric mean divergence measure, or simply weighted AG-divergence”. You can get further information on these divergence measurements in Taneja [28, 33] without weight.

We refer to the measure “G (Ᵽ; Ọ; W) by relative weighted AG-divergence as the measure D(Ᵽ; Ọ; W) by relative weighted J-divergence” for the sake of simplicity.

Dragomir et al. are responsible for the measure D (Ᵽ; Ọ; W) [11]. Numerous authors have investigated the measure F (Ᵽ; Ọ; W) (Shioya and Da-te [22]; Barnet et al. [6]; Lin and Wong [20]). We have here for the first time considered the measure G (Ᵽ; Ọ; W), which is a component of the measure T(Ᵽ; Ọ; W).

Using a different approach than Cressie and Read [8], Kullback-Leibler [19] relative information has only one parametric generalisation, which is given by

$$\varphi_s(\text{Ᵽ};\text{Ǫ}) = \begin{cases} K_s(\text{Ᵽ};\text{Ǫ}) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} \text{ƥ}_i^{s}\, \text{ɋ}_i^{1-s} - 1\right], & s \neq 0,1 \\ K(\text{Ǫ};\text{Ᵽ}) = \sum_{i=1}^{n} \text{ɋ}_i \, log\left(\frac{\text{ɋ}_i}{\text{ƥ}_i}\right) & s = 0 \\ K(\text{Ᵽ};\text{Ǫ}) = \sum_{i=1}^{n} \text{ƥ}_i \, log\left(\frac{\text{ƥ}_i}{\text{ɋ}_i}\right), & s = 1 \end{cases}$$

This, when expanded for weighted distribution, reads as follows:

$$\varphi_s(\text{Ᵽ};\text{Ǫ};W) = \begin{cases} K_s(\text{Ᵽ};\text{Ǫ};W) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} w_i\, \text{ƥ}_i^{s} \text{ɋ}_i^{1-s} - 1\right], & s \neq 0,1 \\ K(\text{Ǫ};\text{Ᵽ};W) = \sum_{i=1}^{n} w_i\, \text{ɋ}_i \, log\left(\frac{\text{ɋ}_i}{\text{ƥ}_i}\right), & s = 0 \\ K(\text{Ᵽ};\text{Ǫ};W) = \sum_{i=1}^{n} w_i\, \text{ƥ}_i \, log\left(\frac{\text{ƥ}_i}{\text{ɋ}_i}\right), & s = 1 \end{cases}$$

(1.14)

for every Ᵽ, Ǫ $\in \Gamma_n$ and s $\in (-\infty, \infty)$.

The measure (1.14) allows for the following specific situations:

(i) $\varphi_{-1}(\text{Ᵽ};\text{Ǫ};W) = \frac{1}{2}\chi^2(\text{Ǫ};\text{Ᵽ};W)$
(ii) $\varphi_0(\text{Ᵽ};\text{Ǫ};W) = K(\text{Ǫ};\text{Ᵽ};W)$
(iii) $\varphi_{1/2}(\text{Ᵽ};\text{Ǫ};W) = 4[1 - B(\text{Ᵽ};\text{Ǫ};W)] = 4h(\text{Ᵽ};\text{Ǫ};W)$
(iv) $\varphi_1(\text{Ᵽ};\text{Ǫ};W) = K(\text{Ᵽ};\text{Ǫ};W)$
(v) $\varphi_2(\text{Ᵽ};\text{Ǫ};W) = \frac{1}{2}\chi^2(\text{Ᵽ};\text{Ǫ};W)$

The weighted measures "B ($\text{Ᵽ};\text{Ǫ};W$) Bhattacharya Coefficient, h ($\text{Ᵽ};\text{Ǫ};W$) and $\chi^2(\text{Ᵽ};\text{Ǫ};W)$" appearing in parts (iii) and (v) above are given by

$$B(\text{Ᵽ};\text{Ǫ};W) = \sum_{i=1}^{n} w_i \sqrt{\text{ƥ}_i \text{ɋ}_i} \quad (1.15)$$

&

$$h(\text{Ᵽ};\text{Ǫ};W) = 1 - B(\text{Ᵽ};\text{Ǫ};W) = \frac{1}{2}\sum_{i=1}^{n} w_i \left(\sqrt{\text{ƥ}_i} - \sqrt{\text{ɋ}_i}\right)^2$$

(1.16)

with condition $\sum_{i=1}^{n} w_i\, \text{ƥ}_i = 1$ and $\sum_{i=1}^{n} w_i\, \text{ɋ}_i = 1$

and

$$\chi^2(Ᵽ; Ọ; W) = \sum_{i=1}^{n} w_i \frac{(\mathfrak{p}_i - \mathfrak{q}_i)^2}{\mathfrak{q}_i} = \sum_{i=1}^{n} w_i \frac{\mathfrak{p}_i^2}{\mathfrak{q}_i} - 1 \tag{1.17}$$

The measures B (Ᵽ; Ọ; W), h (Ᵽ; Ọ; W), and χ^2(Ᵽ; Ọ; W) are known as "weighted Bhattacharya [1] coefficient, weighted Hellinger [17], and weighted Chi-square [21] divergence", respectively.

This paper's goal is to derive bounds on the "relative weighted divergence measures", which we will categories as symmetric weighted divergence measures given by (1.2), (1.10), and (1.11) in terms of generalized "weighted relative information or weighted relative information" of type s given by (1.14).

These bounds are investigated using some Csiszar [5] f-divergence measures.

2 WEIGHTED CSISZAR f-DIVERGENCE

When a convex function is specified, such as f: $[0, \infty) \times (0, \infty) \to \mathcal{R}$ the extension weighted f-divergence measure proposed by Csiszar [5] is given by

$$C_{\mathrm{f}}(Ᵽ; Ọ; W) = \sum_{i=1}^{n} w_i\, \mathfrak{q}_i \mathrm{f}\left(\frac{\mathfrak{p}_i}{\mathfrak{q}_i}\right) \tag{2.1}$$

where Ᵽ, Ọ $\in \Gamma_n$.

The weighted Csiszar function C_{f} (Ᵽ; Ọ; W) is non-negative and convex in the pair of probability distribution Ᵽ, Ọ $\in \Gamma_n \times \Gamma_n$ if f is convex and normal (f (1, 1) = 0). This is well known in the literature [4,5].

Now, we will demonstrate the convexity and non-negativity of the measures listed in section 1.

Illustration 2.1 (Relative Weighted J-Divergence)

Take

$$\mathrm{f}_{\mathcal{D}_1}(\varepsilon; w) = w(\varepsilon - 1)\, log\left(\frac{\varepsilon+1}{2}\right), \varepsilon \in (0, \infty), w > 0 \tag{2.2}$$

in (2.1), then we get C_{f}(Ᵽ; Ọ; W) = $\mathcal{D}$ (Ᵽ; Ọ; W) := $\mathcal{D}_1$

where $\mathcal{D}$ (Ᵽ; Ọ; W) is as given by (1.4).

Differentiating (2.2) w.r.t. ε, we get

$$f'_{\mathcal{D}_1}(\varepsilon; w) = w\left[\frac{(\varepsilon-1)}{\varepsilon+1} + log\left(\frac{\varepsilon+1}{2}\right)\right] \quad \forall\varepsilon \in (0, \infty), w > 0 \tag{2.3}$$

&

$$f''_{\mathcal{D}_1}(\varepsilon; w) = w\frac{(\varepsilon+3)}{(\varepsilon+1)^2} \qquad \forall\varepsilon \in (0, \infty), w > 0 \tag{2.4}$$

Hence, $f_{\mathcal{D}_1}(\varepsilon; w)$ is convex for all ε >0, w>0 since (2.4) shows that $f''_{\mathcal{D}_1}(\varepsilon; w) > 0$ for every ε >0, w > 0. In addition, _$f_{\mathcal{D}_1}$ (1;1)=0 is true. As a result, we can conclude that the relative weighted J-divergence in the pair of probability distributions Ᵽ, Q ∈ $\Gamma_n \times \Gamma_n$ is non-negative and convex.

Illustration 2.2 (Adjoint of relative weighted J-divergence)

Take

$$f_{\mathcal{D}_2}(\varepsilon; w) = w(1-\varepsilon)\, log\left(\frac{\varepsilon+1}{2\varepsilon}\right), \varepsilon \in (0, \infty), w > 0 \tag{2.5}$$

in (2.1), then we get $C_f(Ᵽ; Q; W)$ = D (Q; Ᵽ; W) := D_2 where D (Q; Ᵽ; W) is given by (1.5).

Differentiating (2.5) w.r.t. ε, we get

$$f'_{\mathcal{D}_2}(\varepsilon; w) = w\left[\frac{(\varepsilon-1)}{\varepsilon(\varepsilon+1)} - log\left(\frac{\varepsilon+1}{2\varepsilon}\right)\right] \tag{2.6}$$

and

$$f''_{\mathcal{D}_2}(x; w) = \frac{w(3\varepsilon+1)}{\varepsilon^2(\varepsilon+1)^2}, \varepsilon \in (0, \infty), w > 0 \tag{2.7}$$

Hence, $f_{\mathcal{D}_2}(\varepsilon; w)$is convex for every ε > 0, w > 0, as shown by (2.7), where $f''_{\mathcal{D}_2}(\varepsilon; w) > 0$ for every ε > 0, w > 0. Moreover, $f_{\mathcal{D}_2}$ (1; 1) = 0 is present. As a result, we can conclude that the adjoint of relative "weighted J divergence" in the pair of probability distributions Ᵽ, Q ∈ $\Gamma_n \times \Gamma_n$ is non-negative and convex.

Illustration 2.3 (Relative Weighted JS-divergence)

Take

$$f_{F_1}(\varepsilon; w) = w\left[\frac{1-\varepsilon}{2} - \varepsilon\, log\left(\frac{\varepsilon+1}{2\varepsilon}\right)\right], \varepsilon \in (0, \infty), w > 0 \tag{2.8}$$

in (2.1), then we set $C_f(Ᵽ; Q; W)$ = $F(Ᵽ; Q; W)$:= F_1

where $F(Ᵽ; Q; W)$is given by (1.6)

Differentiating (2.8) w.r.t. ε, we get

$$f'_{F_1}(\varepsilon; w) = w\left[\frac{1-\varepsilon}{2(\varepsilon+1)} - log\left(\frac{\varepsilon+1}{2\varepsilon}\right)\right] \tag{2.9}$$

&

$$f''_{F_1}(\varepsilon; w) = \frac{w}{\varepsilon(\varepsilon+1)^2} \quad \forall\varepsilon \in (0, \infty), w > 0 \tag{2.10}$$

Hence, $f_{F_1}(\varepsilon; w)$is convex for every ε > 0 and w > 0 as shown in (2.10), and for any ε > 0 and w > 0 we observe that $f''_{F_1}(\varepsilon; w)$>0. Also, f_{F_1} (1;1) =0 is present. As a result, we can conclude that the relative weighted JS-divergence in the pair of probability distributions Ᵽ, Q $\in \Gamma_n \times \Gamma_n$ is non-negative and convex.

Illustration 2.4: (Adjoint of relative weighted JS-divergence)

Take

$$f_{F_2}(\varepsilon; w) = w\left[\frac{\varepsilon-1}{2} - log\left(\frac{\varepsilon+1}{2}\right)\right] \quad \forall\varepsilon \in (0, \infty), w > 0 \tag{2.11}$$

in (2.1), then we get $C_f(Ᵽ; Q; W) = F(Q; Ᵽ; W)$: = F_2

where $F(Q; Ᵽ; W)$ is given by (1.8)

Differentiate (2.11) w.r.t. ε, we get

$$f'_{F_2}(\varepsilon; w) = \frac{w(\varepsilon-1)}{2(\varepsilon+1)} \tag{2.12}$$

and

$$f''_{F_2}(\varepsilon; w) = \frac{w}{(\varepsilon+1)^2}, \forall\varepsilon \in (0, \infty), w > 0. \tag{2.13}$$

Since $f''_{F_2}(\varepsilon; w)$> 0 for all ε > 0 and w > 0 as shown by (2.13), $f_{F_2}(\varepsilon; w)$is convex for all ε > 0 and w > 0 as well. Furthermore, f_{F_2} (1; 1) = 0 is present. As a result, we can conclude that the adjoint of relative weighted JS-divergence in the pair of probability distributions Ᵽ, Q $\in \Gamma_n \times \Gamma_n$ is non-negative and convex.

Illustration 2.5 (Relative Weighted AG-divergence).

Take

$$f_{G_1}(\varepsilon; w) = \frac{w(\varepsilon-1)}{2} + \frac{w(\varepsilon+1)}{2} log\left(\frac{\varepsilon+1}{2\varepsilon}\right), \varepsilon \in (0,\infty), w > 0 \quad (2.14)$$

in (2.1), then we get $C_f(Ᵽ; Q; W) = G(Ᵽ; Q; W)$:= G_1 where $G(Ᵽ; Q; W)$is as given by (1.7).

Moreover,

$$f'_{G_1}(\varepsilon; w) = \frac{w}{2}\left[\frac{\varepsilon-1}{\varepsilon} + log\left(\frac{\varepsilon+1}{2\varepsilon}\right)\right] \quad (2.15)$$

and

$$f''_{G_1}(\varepsilon; w) = \frac{w}{2\varepsilon^2(\varepsilon+1)}, \qquad \forall\varepsilon \in (0,\infty), w > 0 \quad (2.16)$$

Thus, $f_{G_1}(\varepsilon; w)$is convex for every ε > 0 and w > 0 as shown by (2.16), where $f''_{G_1}(\varepsilon; w)$ > 0 for every ε > 0 and w > 0. In addition, f_{G_1} (1; 1) = 0 is present. As a result, we can conclude that the relative weighted AG-divergence in the pair of probability distributions Ᵽ, Q $\in \Gamma_n \times \Gamma_n$ is non-negative and convex.

Illustration 2.6 (Adjoint of relative weighted AG-divergence).

Take

$$f_{G_2}(\varepsilon; w) = \frac{w(1-\varepsilon)}{2} + \frac{w(\varepsilon+1)}{2} log\left(\frac{\varepsilon+1}{2}\right), \ \varepsilon \in (0, \infty), w > 0 \quad (2.17)$$

in (2.1), then we get $C_f(Ᵽ; Q; W)$ = $G(Q; Ᵽ; W)$:= G_2, where $G(Q; Ᵽ; W)$is as given by (1.9)

Differentiate (2.17) w.r.t. ε, we get

$$f'_{G_2}(\varepsilon; w) = \frac{w}{2} log\left(\frac{\varepsilon+1}{2}\right) \quad (2.18)$$

&

$$f''_{G_2}(\varepsilon; w) = \frac{w}{2(\varepsilon+1)}, \forall\varepsilon \in (0,\infty), w > 0 \quad (2.19)$$

The fact that$f''_{G_2}(\varepsilon; w)$> 0 for all ε > 0 and w > 0 is evident from equation (2.19), and as a result, $f_{G_2}(\varepsilon; w)$is convex for all ε > 0 and w > 0 And f_{G_2} (1; 1) = 0 is another result. As a result, we can conclude that the adjoint of relative weighted

AG-divergence in the pair of probability distributions Ᵽ, Ꝕ $\in \Gamma_n \times \Gamma_n$ is non-negative and convex.

Illustration 2.7 (Weighted J-divergence).

Take

$$f_J(\varepsilon; w) = w(\varepsilon - 1)\log\varepsilon, \quad \forall\ \varepsilon \in (0, \infty),\ w > 0 \tag{2.20}$$

in (2.1), then we set $C_f(Ᵽ; Ꝕ; W) = J(Ᵽ; Ꝕ; W)$ where $J(Ᵽ; Ꝕ; W)$ is as given by (1.2). Ᵽ, Ꝕ $\in \Gamma_n \times \Gamma_n$

Differentiate (2.20) w.r.t ε, we get

$$f_J'(\varepsilon; w) = w(1 - \varepsilon^{-1} + log\,\varepsilon) \tag{2.21}$$

and

$$f_J''(\varepsilon; w) = \frac{w(\varepsilon+1)}{\varepsilon^2}, \forall\varepsilon \in (0, \infty), w > 0 \tag{2.22}$$

As a result, we can infer from (2.22), that $f_J''(\varepsilon; w) > 0$ for all ε > 0, w > 0, and that $f_J(\varepsilon; w)$ is convex for all ε > 0, w > 0. $f_J(1; 1) = 0$ is another factor. As a result, we can conclude that the weighted J-divergence in the pair of probability distributions Ᵽ, Ꝕ $\in \Gamma_n \times \Gamma_n$ is non-negative and convex.

Illustration 2.8 (Weighted JS-divergence).

Take

$$f_I(\varepsilon; w) = w\frac{\varepsilon}{2} log\,\varepsilon - \frac{w(\varepsilon+1)}{2} log\left(\frac{\varepsilon+1}{2}\right), \forall\varepsilon \in (0, \infty), w>0 \tag{2.23}$$

in (2.1), then we get $C_f(Ᵽ; Ꝕ; W) = I(Ᵽ; Ꝕ; W)$, where $I(Ᵽ; Ꝕ; W)$is as given by (1.10)

Differentiate (2.23) w.r.t. ε, we get

$$f_I'(\varepsilon; w) = -\frac{w}{2} log\left(\frac{\varepsilon+1}{2\varepsilon}\right) \tag{2.24}$$

ands

$$f_I''(\varepsilon; w) = \frac{w}{2\varepsilon(\varepsilon+1)}, \forall\varepsilon \in (0, \infty), w > 0 \tag{2.25}$$

This means that $f_I''(\varepsilon; w) > 0$ for every ε > 0 and that $f_I(\varepsilon; w)$ is convex for every ε > 0 and w > 0 is evident from (2.25). We have f_I (1;1)=0 as well. As a result, we can conclude that the weighted JS-divergence in the pair of probability distributions Ᵽ, Ọ ∈ $\Gamma_n \times \Gamma_n$ is non-negative and convex.

Illustration 2.9 (Weighted AG-divergence):

Take

$$f_T(\varepsilon; w) = w\frac{(\varepsilon+1)}{2} log\left(\frac{\varepsilon+1}{2\sqrt{\varepsilon}}\right),\ \varepsilon \in (0, \infty),\ w>0 \tag{2.26}$$

in (2.1), then we set $C_f(Ᵽ; Ọ; W) = T(Ᵽ; Ọ; W)$

where $T(Ᵽ; Ọ; W)$ is as given by (1.11).

Differentiate (2.26) w.r.t. ε, we get

$$f_T'(\varepsilon; w) = \frac{w}{4}\left[1 - \varepsilon^{-1} + 2\, log\left(\frac{\varepsilon+1}{2\sqrt{\varepsilon}}\right)\right] \tag{2.27}$$

and

$$f_T''(\varepsilon; w) = \frac{w}{4}\left(\frac{\varepsilon^2+1}{\varepsilon^3+\varepsilon^2}\right), \forall \varepsilon \in (0, \infty), w > 0 \tag{2.28}$$

As a result, from (2.28) we may deduce that $f_T''(\varepsilon; w) > 0$ for all ε > 0 and w > 0 and hence. For every ε > 0, w > 0, and $f_T(\varepsilon; w)$ is convex. f_T (1; 1) = 0 is another factor. As a result, we can conclude that the weighted AG-divergence in the pair of probability distributions Ᵽ, Ọ ∈ $\Gamma_n \times \Gamma_n$ is non-negative and convex.

Only the non-negativity and convexity of the "symmetric and non-symmetric weighted divergence measures" are demonstrated in the instances above. Here, we'll use this attribute to obtain bounds for relative information of type s. Refer to Taneja [30] for further information on these measures.

3. CSISZAR-f-DIVERGENCE AND WEIGHTED RELATIVE INFORMATION OF TYPE s

The extensions of Taneja [34] and Taneja and Kumar [28] for weighted distribution are the two theorems that follow.

Theorem 3.1. Let Ᵽ, Ọ ∈ Γ_n and s ∈ $\mathcal{R}$:= (–∞, ∞), then we have

$$0 \le \varphi_s(Ᵽ; Ọ; W) \le E_{\varphi_s}(Ᵽ; Ọ; W) \tag{3.1}$$

here

$$E_{\varphi_s}(Ᵽ; Q; W) = \begin{cases} (s-1)^{-1} \sum_{i=1}^{n} w_i \, (\text{þ}_i - \text{q}_i) \left(\frac{\text{þ}_i}{\text{q}_i}\right)^{s-1}, & s \neq 1 \\ \sum_{i=1}^{n} w_i \, (\text{þ}_i - \text{q}_i) \log\left(\frac{\text{þ}_i}{\text{q}_i}\right), & s = 1 \end{cases} \quad (3.2)$$

Let Ᵽ, Q $\in \Gamma_n$ be such that there exists ℓ, L$\in \mathcal{R}$ with $0 < \ell \leq \frac{\text{þ}_i}{\text{q}_i} \leq \text{L} < \infty$, for every i=1to n, then $0 \leq \varphi_s(Ᵽ; Q; W) \leq A_{\varphi_s}(\ell, \text{L}; W)$ (3.3)

where

$$A_{\varphi_s}(\ell, \text{L}; W) = \frac{w}{4}(\text{L} - \ell)^2 \begin{cases} \frac{\text{L}^{s-1} - \ell^{s-1}}{(\text{L}-\ell)(s-1)}, & s \neq 1 \\ \frac{\log \text{L} - \log \ell}{\text{L} - \ell}, & s = 1 \end{cases} \quad (3.4)$$

Additionally, if we assume that $0 < \ell \leq 1 \leq \text{L} < \infty$, $\ell \neq \text{L}$, then

$$0 \leq \varphi_s(Ᵽ; Q; W) \leq B_{\varphi_s}(\ell, \text{L}; W) \quad (3.5)$$

where

$$B_{\varphi_s}(\ell, \text{L}; W) = \begin{cases} \frac{w[(\text{L}-1)(\ell^s - 1) + (1-\ell)(\text{L}^s - 1)]}{(\text{L}-\ell)s(s-1)}, & s \neq 0,1 \\ w\,\frac{\left[(\text{L}-1)\log\frac{1}{\ell} + (1-\ell)\log\frac{1}{\text{L}}\right]}{(\text{L}-\ell)}, & s = 0 \\ \frac{w[(\text{L}-1)\ell \log \ell + (1-\ell)\text{L} \log \text{L}]}{(\text{L}-\ell)}, & s = 1 \end{cases} \quad (3.6)$$

Moreover, following inequalities hold.

$$E_{\varphi_s}(Ᵽ; Q; W) \leq A_{\varphi_s}(\ell, \text{L}; W) \quad (3.7)$$

$$B_{\varphi_s}(\ell, \text{L}; W) \leq A_{\varphi_s}(\ell, \text{L}; W) \quad (3.8)$$

&

$$B_{\varphi_s}(\ell, \text{L}; W) - \varphi_s(Ᵽ; Q; W) \leq A_{\varphi_s}(\ell, \text{L}; W) \quad (3.9)$$

Theorem 3.2. If f : I × (0, ∞) ⊂ $\mathcal{R}_+$ × (0, ∞) → $\mathcal{R}$ the generating mapping is normalized, with f(1; 1) equal to 0, and it meets the following assumptions:

(i) f can be differentiated twice on $(\ell, L) \times (0, \infty)$

(ii) there exist real constants λ, δ such that $0 < \lambda < \delta$ and

$$\lambda \leq \varepsilon^{2-s} f''(\varepsilon; w) \leq \delta \quad \forall \varepsilon \in (\ell, L), -\infty < s < \infty \tag{3.10}$$

then, we have

$$\lambda \varphi_s(P; Q; W) \leq C_f(P; Q; W) \leq \delta \varphi_s(P; Q; W) \tag{3.11}$$

and

$$\lambda[E_{\varphi_s}(P; Q; W) - \varphi_s(P; Q; W)] \leq E_{C_f}(P; Q; W) - C_f(P; Q; W) \leq \delta[E_{\varphi_s}(P; Q; W) - \varphi_s(P; Q; W)] \tag{3.12}$$

Let P, Q $\in \Gamma_n$ be such that there exist ℓ, L with $0 < \ell \leq \frac{p_i}{q_i} \leq L < \infty$, for every i=1to n then

$$\lambda[A_{\varphi_s}(\ell, L; W) - \varphi_s(P; Q; W)] \leq A_{C_f}(\ell, L; W) - C_f(P; Q; W) \leq \delta[A_{\varphi_s}(\ell, L; W) - \varphi_s(P; Q; W)] \tag{3.13}$$

Further if we suppose that $0 < \ell \leq 1 \leq L < \infty$, $\ell \neq L$ then

$$\lambda[B_{\varphi_s}(\ell, L; W) - \varphi_s(P; Q; W)] \leq B_{C_f}(\ell, L; W) - C_f(P; Q; W) \leq \delta[B_{\varphi_s}(\ell, L; W) - \varphi_s(P; Q; W)] \tag{3.14}$$

By using a few of the Dragomir-related findings [11, 16], the Theorem 3.1 is produced. Several of Dragomir's research findings [12, 14, 15] are brought together by the Theorem 3.2. Referring to Taneja [32] without weighted distribution, Theorem 3.2 has been improved.

Applying Theorem 3.2 in this case entails using various values of f from examples 2.1 to 2.9. This was accomplished by using the inequalities (3.11), and findings for the inequalities (3.12) to (3.14) can be obtained in a manner that is analogous. These specifics are not included here.

4 BOUNDS ON NON-SYMMETRIC WEIGHTED DIVERGENCE MEASURES

In order to set bounds for the measures from (1.4) - (1.9), we have used the condition (3.10) and the inequality (3.11).

Theorem 4.1: The bounds on relative "weighted J-divergence" are as follows:

$$w\frac{\ell^{2-s}(\ell+3)}{(\ell+1)^2}\varphi_s(Ᵽ;Q;W) \le \mathcal{D}(Ᵽ;Q;W) \le w\frac{\mathrm{L}^{2-s}(\mathrm{L}+3)}{(\mathrm{L}+1)^2}\varphi_s(Ᵽ;Q;W), s \le \frac{3}{4} \quad (4.1)$$

&

$$w\frac{\mathrm{L}^{2-s}(\mathrm{L}+3)}{(\mathrm{L}+1)^2}\varphi_s(Ᵽ;Q;W) \le \mathcal{D}(Ᵽ;Q;W) \le \frac{\ell^{2-s}(\ell+3)}{(\ell+1)^2}\varphi_s(Ᵽ;Q;W), \quad s \ge 2 \quad (4.2)$$

Proof: Take

$$g_{\mathcal{D}_1}(\varepsilon;w) = \varepsilon^{2-s}f''_{\mathcal{D}_1}(\varepsilon;w) = \frac{w\varepsilon^{2-s}(\varepsilon+3)}{(\varepsilon+1)^2}, \varepsilon \in (0,\infty), w > 0 \quad (4.3)$$

where $f''_{\mathcal{D}_1}(\varepsilon;w)$ is given by (2.4)

From (4.3), we have

$$g'_{\mathcal{D}_1}(\varepsilon;w) = -\frac{w\varepsilon^{1-s}[(s-1)\varepsilon^2+(4s-3)\varepsilon+3(s-2)]}{(\varepsilon+1)^3}\begin{cases} \ge 0, & s \le \frac{3}{4} \\ \le 0, & s \ge 2 \end{cases} \quad (4.4)$$

From (4.4), we have

$$\lambda = \inf_{\varepsilon\in[\ell,\mathrm{L}]} g_{\mathcal{D}_1}(\varepsilon;w) = \begin{cases} \frac{\ell^{2-s}(\ell+3)}{(\ell+1)^2}, & s \le \frac{3}{4} \\ w\frac{\mathrm{L}^{2-s}(\mathrm{L}+3)}{(\mathrm{L}+1)^2}, & s \ge 2 \end{cases} \quad (4.5)$$

$$\delta = \sup_{\varepsilon\in[\ell,\mathrm{L}]} g_{\mathcal{D}_1}(\varepsilon;w) = \begin{cases} w\frac{\mathrm{L}^{2-s}(\mathrm{L}+3)}{(\mathrm{L}+1)^2}, & s \le \frac{3}{4} \\ \frac{\ell^{2-s}(\ell+3)}{(\ell+1)^2}, & s \ge 2 \end{cases} \quad (4.6)$$

We obtain the inequalities (4.1) and (4.2) in view of (4.5), (4.6), and (3.11).

The following corollary provides a summary of several specific Theorem 4.1 situations.

Corollary 4.1: Bounds that are valid include:

$$w\frac{\ell^3(\ell+3)}{2(\ell+1)^2}\chi^2(Q;P;W) \leq \mathcal{D}(P;Q;W) \leq \frac{wL^3(L+3)}{2(L+1)^2}\chi^2(Q;P;W) \tag{4.7}$$

$$w\frac{\ell^2(\ell+3)}{(\ell+1)^2}K(Q;P;W) \leq \mathcal{D}(P;Q;W) \leq \frac{wL^2(L+3)}{(L+1)^2}K(Q;P;W) \tag{4.8}$$

$$w\frac{4\ell^{3/2}(\ell+3)}{(\ell+1)^2}h(P;Q;W) \leq \mathcal{D}(P;Q;W) \leq \frac{w4L^{1/2}(L+3)}{(L+1)^2}h(P;Q;W) \tag{4.9}$$

&

$$w\frac{(L+3)}{2(L+1)^2}\chi^2(P;Q;W) \leq \mathcal{D}(P;Q;W) \leq \frac{w(\ell+3)}{2(\ell+1)^2}\chi^2(P;Q;W), \tag{4.10}$$

Proof: By assuming that s = 1, s = 0, and s = 1/2, respectively, inequality (4.7), (4.8), and (4.9) result from (4.1). By assuming that s = 2, the inequality (4.10) emerges from (4.2).

The inequalities (4.1) and (4.2) do not include the situation s = 1. This is what the proposition that follows will do individually.

Proposition 4.1: The next inequality is true.:

$$\mathcal{D}(P;Q;W) \leq \frac{9}{8}K(P;Q;W) \tag{4.11}$$

Proof: For s = 1 in (4.3) we get

$$g_{\mathcal{D}_1}(\varepsilon;w) = \frac{w\varepsilon(\varepsilon+3)}{(\varepsilon+1)^2}, \forall\varepsilon \in (0,\infty), w > 0 \tag{4.12}$$

This gives

$$g'_{\mathcal{D}_1}(\varepsilon;w) = -\frac{w(\varepsilon-3)}{(\varepsilon+1)^3}\begin{cases}\geq 0, & \varepsilon \leq 3\\ \leq 0, & \varepsilon \geq 3\end{cases} \tag{4.13}$$

Thus, we deduce from (4.13) that the function $g_{\mathcal{D}_1}$ $(\varepsilon; w)$ given by (4.12) is rising in the range of ε ∈(0, 3) and decreasing in the range of ε ∈(3, ∞), and as a result

$$\delta = \sup_{\varepsilon\in(0,\infty)} g_{\mathcal{D}_1}(\varepsilon; w) = g_{\mathcal{D}_1}(3,1) = \frac{9}{8} \tag{4.14}$$

Now, (4.14) and (3.11) work together to get the desired outcome.

Theorem 4.2. The bounds on "weighted relative J-divergence" adjoint are as follows:

$$w\frac{\ell^{-s}(3\ell+1)}{(\ell+1)^2}\varphi_s(\text{Ᵽ}; \text{Ọ}; W) \le \mathcal{D}(\text{Ọ}; \text{Ᵽ}; W) \le \frac{w\text{L}^{-s}(3\text{L}+1)}{(\text{L}+1)^2}\varphi_s(\text{Ᵽ}; \text{Ọ}; W), \quad s \le -1 \tag{4.15}$$

&

$$\frac{w\text{L}^{-s}(3\text{L}+1)}{(\text{L}+1)^2}\varphi_s(\text{Ᵽ}; \text{Ọ}; W) \le \mathcal{D}(\text{Ọ}; \text{Ᵽ}; W) \le w\frac{\ell^{-s}(3\ell+1)}{(\ell+1)^2}\varphi_s(\text{Ᵽ}; \text{Ọ}; W), \quad s \ge \frac{1}{4} \tag{4.16}$$

Proof: Take

$$g_{\mathcal{D}_2}(\varepsilon; w) = \varepsilon^{2-s}\text{f}''_{\mathcal{D}_2}(\varepsilon; w) = \frac{w\varepsilon^{-s}(3\varepsilon+1)}{(\varepsilon+1)^2}, \varepsilon \in (0,\infty), w > 0 \tag{4.17}$$

where $\text{f}''_{\mathcal{D}_2}(\varepsilon; w)$ is given by (2.7)

From (4.17), we can get

$$g'_{\mathcal{D}_2}(\varepsilon; w) = -\frac{w\varepsilon^{-s}[3(s+1)\varepsilon^2+(4s-1)\varepsilon+s]}{(\varepsilon+1)^3}\begin{cases}\ge 0, & s \le -1\\ \le 0, & s \ge \frac{1}{4}\end{cases} \tag{4.18}$$

In view of (4.18), we get

$$\lambda = \inf_{\varepsilon\in[\ell,\mathrm{L}]} g_{\mathcal{D}_2}(\varepsilon; w) = \min_{\varepsilon\in[\ell,\mathrm{L}]} g_{\mathcal{D}_2}(\varepsilon; w) = \begin{cases} w\dfrac{\ell^{-s}(3\ell+1)}{(\ell+1)^2}, & s \le -1 \\ \dfrac{w\mathrm{L}^{-s}(3\mathrm{L}+1)}{(\mathrm{L}+1)^2}, & s \ge \dfrac{1}{4} \end{cases}$$

(4.19)

$$\delta = \sup_{\varepsilon\in[\ell,\mathrm{L}]} g_{\mathcal{D}_2}(\varepsilon; w) = \begin{cases} \dfrac{w\mathrm{L}^{-s}(3\mathrm{L}+1)}{(\mathrm{L}+1)^2}, s \le -1 \\ w\dfrac{\ell^{-s}(3\ell+1)}{(\ell+1)^2}, s \ge \dfrac{1}{4} \end{cases}$$

(4.20)

We now obtain the inequalities (4.15) and (4.11) from (4.19), (4.20), and (3.11). The following corollary provides a summary of several specific examples of Theorem 4.2.

Corollary 4.2: Bounds that are valid include:

$$\frac{w\ell(3\ell+1)}{2(\ell+1)^2}\chi^2(Q;P;W) \le \mathcal{D}(Q;P;W) \le \frac{w\mathrm{L}(3\mathrm{L}+1)}{2(\mathrm{L}+1)^2}\chi^2(Q;P;W)$$

(4.21)

$$\frac{4w(3\mathrm{L}+1)}{\sqrt{\mathrm{L}}(\mathrm{L}+1)^2}h(P;Q;W) \le \mathcal{D}(Q;P;W) \le \frac{4w(3\ell+1)}{\sqrt{\ell}(\ell+1)^2}h(P;Q;W)$$

(4.22)

$$\frac{w(3\mathrm{L}+1)}{\mathrm{L}(\mathrm{L}+1)^2}K(P;Q;W) \le \mathcal{D}(Q;P;W) \le \frac{w(3\ell+1)}{\ell(\ell+1)^2}K(P;Q;W)$$

(4.23)

$$\frac{w(3\mathrm{L}+1)}{2\mathrm{L}^2(\mathrm{L}+1)^2}\chi^2(P;Q;W) \le \mathcal{D}(Q;P;W) \le \frac{w(3\ell+1)}{2\ell^2(\ell+1)^2}\chi^2(P;Q;W)$$

(4.24)

Proof: By setting s = 1, inequality (4.21) derives from (4.15). By assuming that s = 1/2, s = 1, and s = 2 accordingly, the inequality (4.22), (4.23), and (4.24) emerge from (4.16).

The inequalities (4.15) and (4.16) do not include the situation s = 0 .This is what the proposition that follows will do individually.

Proposition 4.2: The next inequality is true.:

$$\mathcal{D}(Q;P;W) \leq \frac{9}{8}\boldsymbol{K}(Q;P;W) \qquad (4.25)$$

Proof: For s = 0 in (4.17) we have

$$\boldsymbol{g}_{\mathcal{D}_2}(\varepsilon;w) = \frac{\boldsymbol{w}(\mathbf{3}\varepsilon+\mathbf{1})}{(\varepsilon+\mathbf{1})^{\mathbf{2}}}, \forall\varepsilon \in (\mathbf{0},\infty), \boldsymbol{w} > \mathbf{0} \qquad (4.26)$$

This gives

$$\boldsymbol{g}'_{\mathcal{D}_2}(\varepsilon;w) = -\frac{\boldsymbol{w}(\mathbf{3}\varepsilon-\mathbf{1})}{(\varepsilon+\mathbf{1})^{\mathbf{3}}}\begin{cases}\geq \mathbf{0}, \varepsilon \leq \frac{\mathbf{1}}{\mathbf{3}} \\ \leq \mathbf{0}, \varepsilon \geq \frac{\mathbf{1}}{\mathbf{3}}\end{cases} \qquad (4.27)$$

Thus, we deduce from (4.27) that the function $\boldsymbol{g}_{\mathcal{D}_2}(\varepsilon;w)$ provided by (4.26) is growing in $\varepsilon \in \left(\mathbf{0},\frac{\mathbf{1}}{\mathbf{3}}\right)$ and decreasing in $\varepsilon \in \left(\frac{\mathbf{1}}{\mathbf{3}},\infty\right)$, and hence

$$\delta = \sup_{\varepsilon\in(0,\infty)} \boldsymbol{g}_{\mathcal{D}_2}(\varepsilon;w) = \boldsymbol{g}_{\mathcal{D}_2}\left(\frac{\mathbf{1}}{\mathbf{3}},\mathbf{1}\right) = \frac{\mathbf{9}}{\mathbf{8}} \qquad (4.28)$$

Now, (4.28) and (3.11) work together to get the desired outcome.

Theorem 4.3 The bounds on relative "weighted JS-divergence" are as follows:

$$\frac{w\ell^{1-s}}{(\ell+1)^2}\varphi_s(P;Q;W) \leq F(P;Q;W) \leq \frac{w\mathrm{L}^{1-s}}{(\mathrm{L}+1)^2}\varphi_s(P;Q;W), s \leq -1 \qquad (4.29)$$

&

$$\frac{w\mathrm{L}^{1-s}}{(\mathrm{L}+1)^2}\varphi_s(P;Q;W) \leq F(P;Q;W) \leq \frac{w\ell^{1-s}}{(\ell+1)^2}\varphi_s(P;Q;W), s \geq 1 \qquad (4.30)$$

Proof: Take

$$g_{F_1}(\varepsilon;w) = \varepsilon^{2-s}\mathrm{f}''_{F_1}(\varepsilon;w) = \frac{w\varepsilon^{1-s}}{(\varepsilon+1)^2}, \varepsilon \in (0,\infty), w > 0 \qquad (4.31)$$

where $\mathrm{f}''_{F_1}(\varepsilon;w)$ is given by (2.10)

From (4.31) we have

$$g_{F_1}^{'}(\varepsilon; w) = -\frac{w\varepsilon^{-s}[(s+1)\varepsilon+(s-1)]}{(\varepsilon+1)^3}\begin{cases}\geq 0, s \leq -1\\ \leq 0, s \geq 1\end{cases} \tag{4.32}$$

From (4.32), we get

$$\lambda = \inf_{\varepsilon\in[\ell,L]} g_{F_1}(\varepsilon; w)\begin{cases}\frac{w\ell^{1-s}}{(\ell+1)^2}, s \leq -1\\ \frac{wL^{1-s}}{(L+1)^2}, s \geq 1\end{cases} \tag{4.33}$$

$$\delta = \sup_{\varepsilon\in[\ell,L]} g_{F_1}(\varepsilon; w) = \begin{cases}\frac{wL^{1-s}}{(L+1)^2}, s \leq -1\\ \frac{w\ell^{1-s}}{(\ell+1)^2}, s \geq 1\end{cases} \tag{4.34}$$

Now, (3.11) when combined with (4.33) and (4.34) produce the desired outcome.

The following corollary provides a summary of several specific examples of Theorem 4.3.

Corollary 4.3 Bounds that are valid include:

$$\frac{w\ell^2}{2(\ell+1)^2}\chi^2(Q; P; W) \leq F(P; Q; W) \leq \frac{wL^2}{2(L+1)^2}\chi^2(Q; P; W) \tag{4.35}$$

$$\frac{w}{(L+1)^2}K(P; Q; W) \leq F(P; Q; W) \leq \frac{w}{(\ell+1)^2}K(P; Q; W) \tag{4.36}$$

&

$$\frac{w}{2L(L+1)^2}\chi^2(P; Q; W) \leq F(P; Q; W) \leq \frac{w}{2\ell(\ell+1)^2}\chi^2(P; Q; W) \tag{4.37}$$

Proof: By setting s = 1, inequality (4.35) follows from (4.29). By using s = 1 and s = 2, respectively, the inequality (4.36) and (4.37) follows from (4.30).

The inequalities (4.29) and (4.30) do not include the instances s = 0 and s=1/2. They will each be addressed separately in the statement that follows.

Proposition 4.3: The next inequality is true.:

$$F(P; Q; W) \leq \frac{1}{4}K(Q; P; W) \tag{4.38}$$

&

$$\mathrm{F}(\text{Ᵽ}; \text{Ϙ}; W) \leq \frac{3\sqrt{3}}{4} h(\text{Ᵽ}; \text{Ϙ}; W) \tag{4.39}$$

Proof: **Case I:** When s = 0 in (4.31) we have

$$g_{F_1}(\varepsilon; w) = \frac{w\varepsilon}{(\varepsilon+1)^2}, \forall \varepsilon \in (0, \infty), w > 0 \tag{4.40}$$

This gives

$$g'_{F_1}(\varepsilon; w) = -\frac{w(\varepsilon-1)}{(\varepsilon+1)^3} \begin{cases} \geq 0, \varepsilon \leq 1 \\ \leq 0, \varepsilon \geq 1 \end{cases} \quad w > 0 \tag{4.41}$$

The function $g_{F_1}(\varepsilon; w)$ given by (4.41) is growing in $\varepsilon \in (0, 1)$ and decreasing in $\varepsilon \in (1, \infty)$, as shown in (4.41), and as a result

$$\delta = \sup_{\varepsilon \in (0,\infty)} g_{F_1}(\varepsilon; w) = g_{F_1} = \frac{1}{4} \tag{4.42}$$

Now, (4.42) and (3.11) produce the inequality (4.38).

Case II: When $s = \frac{1}{2}$ in (4.31), we have

$$g_{F_1}(\varepsilon; w) = \frac{w\sqrt{\varepsilon}}{(\varepsilon+1)^2}, \forall \varepsilon \in (0, \infty), w > 0 \tag{4.43}$$

This gives

$$g'_{F_1}(\varepsilon; w) = -\frac{w(3\varepsilon-1)}{2\sqrt{\varepsilon}(\varepsilon+1)^3} \begin{cases} \geq 0, \varepsilon \leq \frac{1}{3} \\ \leq 0, \varepsilon \geq \frac{1}{3} \end{cases}. \tag{4.44}$$

Inferring that the function $g_{F_1}(\varepsilon; w)$ given by (4.43) is rising in $\varepsilon \in \left(0, \frac{1}{3}\right)$ and decreasing in $\varepsilon \in \left(\frac{1}{3}, \infty\right)$ from (4.44), we then deduce that

$$\delta = \sup_{\varepsilon \in (0,\infty)} g_{F_1}(\varepsilon; w) = g_{F_1}\left(\frac{1}{3}, 1\right) = \frac{3\sqrt{3}}{16} \tag{4.45}$$

Currently, (4.45) and (3.11) present the inequalities (4.39).

Theorem 4.4 The bounds on adjoint of relative "weighted JS-divergence" are as follows:

$$\frac{w\ell^{2-s}}{(\ell+1)^2} \varphi_s(\text{Ᵽ}; \text{Ϙ}; W) \leq F(\text{Ϙ}; \text{Ᵽ}; W) \leq \frac{w\mathrm{L}^{2-s}}{(\mathrm{L}+1)^2} \varphi_s(\text{Ᵽ}; \text{Ϙ}; W), s \leq 0 \tag{4.46}$$

&

$$\frac{w\mathrm{L}^{2-s}}{(\mathrm{L}+1)^2}\varphi_s(Ᵽ;Q;W) \leq F(Q;Ᵽ;W) \leq \frac{w\ell^{2-s}}{(\ell+1)^2}\varphi_s(Ᵽ;Q;W), s \geq 2 \qquad (4.47)$$

Proof: Take

$$g_{F_2}(\varepsilon;w) = \varepsilon^{2-s}f''_{F_2}(\varepsilon;w) = \frac{w\varepsilon^{2-s}}{(\varepsilon+1)^2}, \varepsilon \in (0,\infty), w > 0 \qquad (4.48)$$

where $f''_{F_2}(\varepsilon;w)$ is as given by (2.13)

From (4.48) we have

$$g'_{F_2}(\varepsilon;w) = -\frac{w\varepsilon^{1-s}[s\varepsilon+(s-2)]}{(\varepsilon+1)^3}\begin{cases}\geq 0, s \leq 0\\ \leq 0, s \geq 2\end{cases} \qquad (4.49)$$

From (4.49), we have

$$\lambda = \inf_{\varepsilon\in[\ell,\mathrm{L}]} g_{F_2}(\varepsilon;w) = \begin{cases}\frac{w\ell^{2-s}}{(\ell+1)^2}, s \leq 0\\ \frac{w\mathrm{L}^{2-s}}{(\mathrm{L}+1)^2}, s \geq 2\end{cases} \qquad (4.50)$$

$$\delta = \sup_{\varepsilon\in[\ell,\mathrm{L}]} g_{F_2}(\varepsilon;w) = \begin{cases}\frac{w\mathrm{L}^{2-s}}{(\mathrm{L}+1)^2}, s \leq 0\\ \frac{w\ell^{2-s}}{(\ell+1)^2}, s \geq 2\end{cases} \qquad (4.51)$$

Now, (4.50) and (4.51), when combined with (3.11), produce the desired outcome.

The following corollary summarizes several specific instances of the Theorem 4.4:

Corollary 4.4: The following bounds hold:

$$\frac{w\ell^3}{2(\ell+1)^2}\chi^2(Q;Ᵽ;W) \leq F(Q;Ᵽ;W) \leq \frac{w\mathrm{L}^3}{2(\mathrm{L}+1)^2}\chi^2(Q;Ᵽ;W) \qquad (4.52)$$

$$\frac{w\ell^2}{(\ell+1)^2}K(Q;Ᵽ;W) \leq F(Q;Ᵽ;W) \leq \frac{w\mathrm{L}^2}{(\mathrm{L}+1)^2}K(Q;Ᵽ;W) \qquad (4.53)$$

and

$$\frac{w}{2(\mathrm{L}+1)^2}\chi^2(Ᵽ;Q;W) \leq F(Q;Ᵽ;W) \leq \frac{w}{2(\ell+1)^2}\chi^2(Ᵽ;Q;W) \qquad (4.54)$$

Inequalities (4.52) and (4.53) follow from (4.46) by taking s = –1 and s = 0 respectively. The inequalities (4.54) follow from (4.47) by taking s = 2.

The cases $s=\frac{1}{2}$ and s = 1 are not included in the inequalities (4.46) and (4.47). This we shall do separately in the following proposition.

Proposition 4.4: The next inequality is true.:

$$F(Ọ;Ᵽ;W) \leq \frac{3\sqrt{3}}{4} h(Ᵽ;Ọ;W) \tag{4.55}$$

&

$$F(Ọ;Ᵽ;W) \leq \frac{1}{4} K(Ᵽ;Ọ;W) \tag{4.56}$$

Proof: Case I: When $s=\frac{1}{2}$ in (4.48), we have

$$g_{F_2}(\varepsilon;w) = \frac{w\varepsilon\sqrt{\varepsilon}}{(\varepsilon+1)^2}, \forall\varepsilon \in (0,\infty), w > 0. \tag{4.57}$$

This gives

$$g'_{F_2}(\varepsilon;w) = -\frac{w\sqrt{\varepsilon}(\varepsilon-3)}{2(\varepsilon+1)^3}\begin{cases}\geq 0, \varepsilon \leq 3\\ \leq 0, \varepsilon \geq 3\end{cases} \tag{4.58}$$

As a result of (4.58), we deduce that the values of the function $g_{F_2}(\varepsilon;w)$ given by (4.57) are growing in ε ∈ (0, 3) and decreasing in ε ∈ (3, ∞), and as a result

$$\delta = \sup_{\varepsilon\in(0,\infty)} g_{F_2}(\varepsilon;w) = g_{F_2}(3;1) = \frac{3\sqrt{3}}{16} \tag{4.59}$$

Now (4.59) together with (3.11) gives the inequality (4.55)

Case II: When s = 1, in (4.48), we have

$$g_{F_2}(\varepsilon;w) = \frac{w\varepsilon}{(\varepsilon+1)^2}, \forall\varepsilon \in (0,\infty), w > 0 \tag{4.60}$$

This gives $g'_{F_2}(\varepsilon;w) = -\frac{w(\varepsilon-1)}{(\varepsilon+1)^3}\begin{cases}\geq 0, \varepsilon \leq 1\\ \leq 0, \varepsilon \geq 1\end{cases}$ (4.61)

The function $g_{F_2}(\varepsilon;w)$ presented in (4.60) is therefore growing in ε ∈ (0, 1) and decreasing in ε ∈ (1, ∞), as shown by (4.61).So

$$\delta = \sup_{\varepsilon\in(0,\infty)} g_{F_2}(\varepsilon; w) = g_{F_2}(1;1) = \frac{1}{4} \qquad (4.62)$$

Now, (4.62) and (3.11) produce the inequality (4.56).

Theorem 4.5: The bounds on relative "weighted AG-divergence" are as follows:.

$$\frac{w}{2\ell^s(\ell+1)}\varphi_s(Ᵽ; Q; W) \leq G(Ᵽ; Q; W) \leq \frac{w}{2L^s(L+1)}\varphi_s(Ᵽ; Q; W), s \leq -1 \qquad (4.63)$$

&

$$\frac{w}{2L^s(L+1)}\varphi_s(Ᵽ; Q; W) \leq G(Ᵽ; Q; W) \leq \frac{w}{2\ell^s(\ell+1)}\varphi_s(Ᵽ; Q; W), s \geq 0 \qquad (4.64)$$

Proof: Take

$$g_{G_1}(\varepsilon; w) = \varepsilon^{2-s} f''_{G_1}(\varepsilon; w) = \frac{w\varepsilon^{-s}}{2(\varepsilon+1)}, \varepsilon \in (0,\infty), w > 0 \qquad (4.65)$$

where $f''_{G_1}(\varepsilon; w)$ is as given by (2.16)

From (4.65) we have

$$g'_{G_1}(\varepsilon; w) = -\frac{w\varepsilon^{-1-s}[(s+1)\varepsilon+s]}{2(\varepsilon+1)^2}\begin{cases}\geq 0, s \leq -1\\ \leq 0, s \geq 0\end{cases}. \qquad (4.66)$$

From (4.66), we get

$$\lambda = \inf_{\varepsilon\in[\ell,L]} g_{G_1}(\varepsilon; w) = \begin{cases}\frac{w}{2\ell^s(\ell+1)}, s \leq -1\\ \frac{w}{2L^s(L+1)}, s \geq 0\end{cases} \qquad (4.67)$$

$$\delta = \sup_{\varepsilon\in[\ell,L]} g_{G_1}(\varepsilon; w) = \begin{cases}\frac{w}{2L^s(L+1)}, s \leq -1\\ \frac{w}{2\ell^s(\ell+1)}, s \geq 0\end{cases}$$

Now, (4.67), (4.68), and (3.11) work together to provide the desired outcome.

The following corollary provides a summary of several specific examples of the Theorem 4.5.

Corollary 4.5: The following bounds hold:

$$\frac{w\ell}{4(\ell+1)}\chi^2(Q;P;W) \leq G(P;Q;W) \leq \frac{wL}{4(L+1)}\chi^2(Q;P;W) \quad (4.69)$$

$$\frac{w}{2(L+1)}K(Q;P;W) \leq G(P;Q;W) \leq \frac{w}{2(\ell+1)}K(Q;P;W) \quad (4.70)$$

$$\frac{2w}{\sqrt{L}(L+1)}h(P;Q;W) \leq G(P;Q;W) \leq \frac{2w}{\sqrt{\ell}(\ell+1)}h(P;Q;W) \quad (4.71)$$

$$\frac{w}{2L(L+1)}K(P;Q;W) \leq G(P;Q;W) \leq \frac{w}{2\ell(\ell+1)}K(P;Q;W) \quad (4.72)$$

&

$$\frac{w}{4L^2(L+1)}\chi^2(P;Q;W) \leq G(P;Q;W) \leq \frac{w}{4\ell^2(\ell+1)}\chi^2(P;Q;W) \quad (4.73)$$

Proof: By setting s = 1, the inequality (4.69) follows from (4.63). By assuming that s = 0, s=1/2, s=1, and s = 2 respectively, the inequality (4.70), (4.71), (4.72), and (4.73) emerge from (4.64).

Theorem 4.6: The bounds on adjoint of relative "weighted AG-divergence" are valid as follows:

$$\frac{w\ell^2}{2\ell^s(\ell+1)}\varphi_s(P;Q;W) \leq G(Q;P;W) \leq \frac{wL^2}{2L^s(L+1)}\varphi_s(P;Q;W), s \leq 1 \quad (4.74)$$

$$\frac{wL^2}{2L^s(L+1)}\varphi_s(P;Q;W) \leq G(Q;P;W) \leq \frac{w\ell^2}{2\ell^s(\ell+1)}\varphi_s(P;Q;W), s \geq 2 \quad (4.75)$$

Proof: Take

$$g_{G_2}(\varepsilon;w) = \varepsilon^{2-s}f''_{G_2}(\varepsilon;w) = \frac{w\varepsilon^{2-s}}{2(\varepsilon+1)}, \varepsilon \in (0,\infty), w > 0 \quad (4.76)$$

where $f''_{G_2}(\varepsilon;w)$ is given by (2.19).

From (4.76) we have

$$g'_{G_2}(\varepsilon;w) = -\frac{w\varepsilon^{1-s}[(s-1)\varepsilon+(s-2)]}{2(\varepsilon+1)^2}\begin{cases}\geq 0, s \leq 1\\ \leq 0, s \geq 2\end{cases} \quad (4.77)$$

From (4.77) we get

$$\lambda = \inf_{\varepsilon \in [\ell, L]} g_{G_2}(\varepsilon; w) = \begin{cases} \frac{w\ell^2}{2\ell^s(\ell+1)}, s \leq 1 \\ \frac{wL^2}{2L^s(L+1)}, s \geq 2 \end{cases} \quad (4.78)$$

$$\delta = \sup_{\varepsilon \in [\ell, L]} g_{G_2}(\varepsilon; w) = \begin{cases} \frac{wL^2}{2L^s(L+1)}, s \leq 1 \\ \frac{w\ell^2}{2\ell^s(\ell+1)}, s \geq 2 \end{cases} \quad (4.79)$$

Now, 4.78, 4.79, and 3.11 work together to provide the desired outcome.

This corollary summarizes some specific instances of Theorem 4.6:

Corollary 4.6: The following bounds are valid:

$$\frac{w\ell^3}{4(\ell+1)} \chi^2(Q; Ᵽ; W) \leq G(Q; Ᵽ; W) \leq \frac{wL^3}{4(L+1)} \chi^2(Q; Ᵽ; W) \quad (4.80)$$

$$\frac{w\ell^2}{2(\ell+1)} K(Q; Ᵽ; W) \leq G(Q; Ᵽ; W) \leq \frac{wL^2}{2(L+1)} K(Q; Ᵽ; W) \quad (4.81)$$

$$\frac{2w\ell\sqrt{\ell}}{(\ell+1)} h(Ᵽ; Q; W) \leq G(Q; Ᵽ; W) \leq \frac{2wL\sqrt{L}}{(L+1)} h(Ᵽ; Q; W) \quad (4.82)$$

$$\frac{w\ell}{2(\ell+1)} K(Ᵽ; Q; W) \leq G(Q; Ᵽ; W) \leq \frac{wL}{2(L+1)} K(Ᵽ; Q; W) \quad (4.83)$$

$$\frac{w}{4(L+1)} \chi^2(Ᵽ; Q; W) \leq G(Q; Ᵽ; W) \leq \frac{w}{4(\ell+1)} \chi^2(Ᵽ; Q; W) \quad (4.84)$$

Proof: By assuming that s = 1, s = 0, s=1/2, and s = 1 correspondingly, inequality (4.80), (4.81), (4.82), and (4.83) arise from (4.74). By assuming that s = 2, the inequality (4.84) follows from (4.75).

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